Josephson junctions in a magnetic field: insights from coupled pendula

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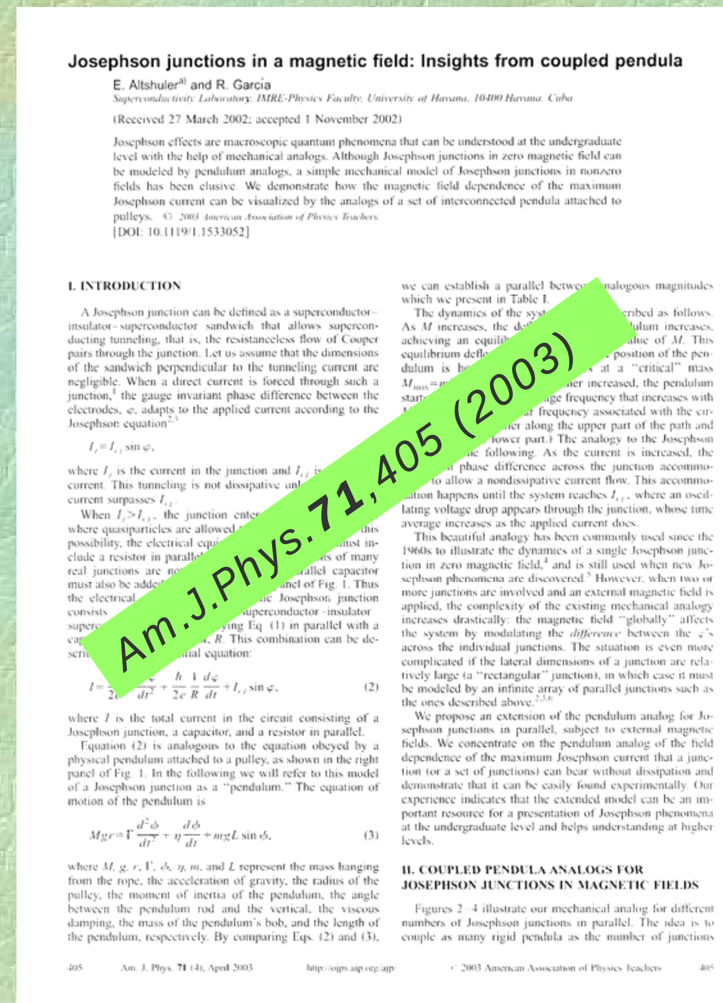

Josephson junctions in a magnetic field: Insights from coupled pendula

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Josephson effects are macroscopic quantum phenomena that can be understood at the undergraduate level with the help of mechanical analogs. Although Josephson junctions in zero magnetic field can be modeled by pendulum analogs, a simple mechanical model of Josephson junctions in nonzero fields has been elusive. We demonstrate how the magnetic field dependence of the maximum Josephson current can be visualized by the analogs of a set of interconnected pendula attached to pulleys. © 2003 American Association of Physics Teachers.

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I. INTRODUCTION

A Josephson junction can be defined as a superconductor–insulator–superconductor sandwich that allows superconducting tunneling, that is, the resistanceless flow of Cooper pairs through the junction. Let us assume that the dimensions of the sandwich perpendicular to the tunneling current are negligible. When a direct current is forced through such a junction,[1] the gauge invariant phase difference between the electrodes, φ, adapts to the applied current according to the Josephson equation[2,3]

$$I_j = I_{cj} \sin \varphi,$$

where I_j is the current in the junction and I_{cj} is [illegible] current. This tunneling is not dissipative unl[illegible] current surpasses I_{cj}.

When $I_j > I_{cj}$, the junction enter[illegible] where quasiparticles are allowed [illegible] this possibility, the electrical equi[illegible] must include a resistor in parall[illegible]s of many real junctions are no[illegible]rallel capacitor must also be adde[illegible]anel of Fig. 1. Thus the electrical [illegible]ic Josephson junction consists [illegible] superconductor–insulator superc[illegible]ying Eq. (1) in parallel with a ca[illegible], R. This combination can be descri[illegible]ial equation:

$$I = [illegible] \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e}\frac{1}{R}\frac{d\varphi}{dt} + I_{cj}\sin\varphi, \quad (2)$$

where I is the total current in the circuit consisting of a Josephson junction, a capacitor, and a resistor in parallel.

Equation (2) is analogous to the equation obeyed by a physical pendulum attached to a pulley, as shown in the right panel of Fig. 1. In the following we will refer to this model of a Josephson junction as a “pendulum.” The equation of motion of the pendulum is

$$Mgr = \Gamma \frac{d^2\phi}{dt^2} + \eta \frac{d\phi}{dt} + mgL \sin\phi, \quad (3)$$

where M, g, r, Γ, ϕ, η, m, and L represent the mass hanging from the rope, the acceleration of gravity, the radius of the pulley, the moment of inertia of the pendulum, the angle between the pendulum rod and the vertical, the viscous damping, the mass of the pendulum's bob, and the length of the pendulum, respectively. By comparing Eqs. (2) and (3), we can establish a parallel betwe[illegible]nalogous magnitudes which we present in Table I.

The dynamics of the syst[illegible]cribed as follows. As M increases, the de[illegible]dulum increases, achieving an equili[illegible]alue of M. This equilibrium defle[illegible] position of the pendulum is h[illegible] at a “critical” mass M_{max}=[illegible]her increased, the pendulum start[illegible]ge frequency that increases with [illegible]r frequency associated with the cir[illegible]er along the upper part of the path and [illegible] lower part.) The analogy to the Josephson [illegible]e following. As the current is increased, the [illegible] phase difference across the junction accommo[illegible] to allow a nondissipative current flow. This accommo[illegible]ation happens until the system reaches I_{cj}, where an oscillating voltage drop appears through the junction, whose time average increases as the applied current does.

This beautiful analogy has been commonly used since the 1960s to illustrate the dynamics of a single Josephson junction in zero magnetic field,[4] and is still used when new Josephson phenomena are discovered.[5] However, when two or more junctions are involved and an external magnetic field is applied, the complexity of the existing mechanical analogy increases drastically: the magnetic field “globally” affects the system by modulating the *difference* between the φ's across the individual junctions. The situation is even more complicated if the lateral dimensions of a junction are relatively large (a “rectangular” junction), in which case it must be modeled by an infinite array of parallel junctions such as the ones described above.[2,3,6]

We propose an extension of the pendulum analog for Josephson junctions in parallel, subject to external magnetic fields. We concentrate on the pendulum analog of the field dependence of the maximum Josephson current that a junction (or a set of junctions) can bear without dissipation and demonstrate that it can be easily found experimentally. Our experience indicates that the extended model can be an important resource for a presentation of Josephson phenomena at the undergraduate level and helps understanding at higher levels.

II. COUPLED PENDULA ANALOGS FOR JOSEPHSON JUNCTIONS IN MAGNETIC FIELDS

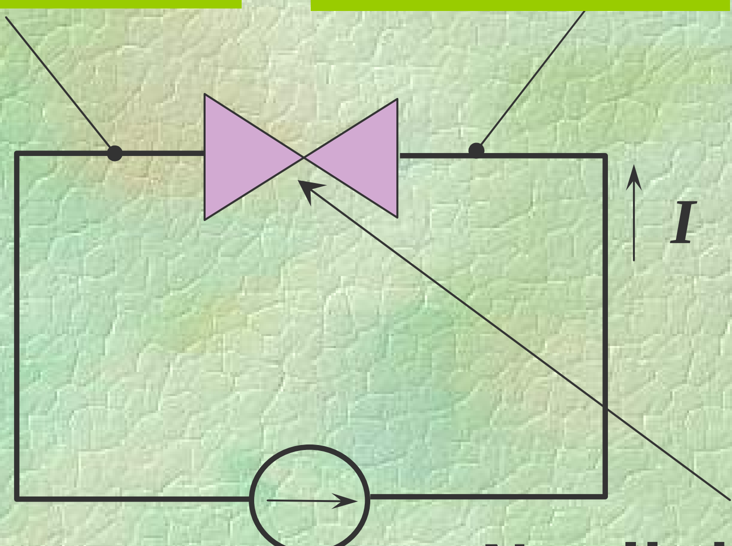
Figures 2–4 illustrate our mechanical analog for different numbers of Josephson junctions in parallel. The idea is to couple as many rigid pendula as the number of junctions

What is a “concentrated” Josephson junction?

$$I_j = I_{cj} \sin \varphi$$

$$\psi_{SC2} \sim e^{-i\theta_2}$$

$$\psi_{SC1} \sim e^{-i\theta_1}$$



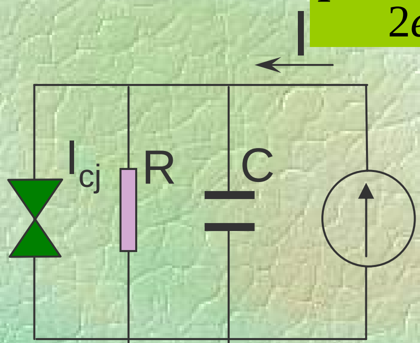
where

$$\varphi = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0} \int_{SC1}^{SC2} \vec{A} \cdot d\vec{l}$$

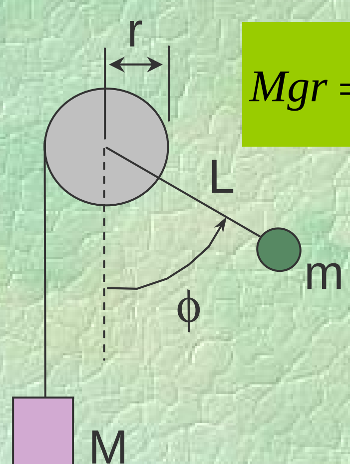
Negligible dimensions perpendicular to current flow

Modelling a single, concentrated Josephson junction

$$I = \frac{\hbar}{2e} C \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e}\frac{1}{R}\frac{d\varphi}{dt} + I_{cj}\sin\varphi$$



$$Mgr = \Gamma \frac{d^2\phi}{dt^2} + \eta\frac{d\phi}{dt} + mgL\sin\phi$$

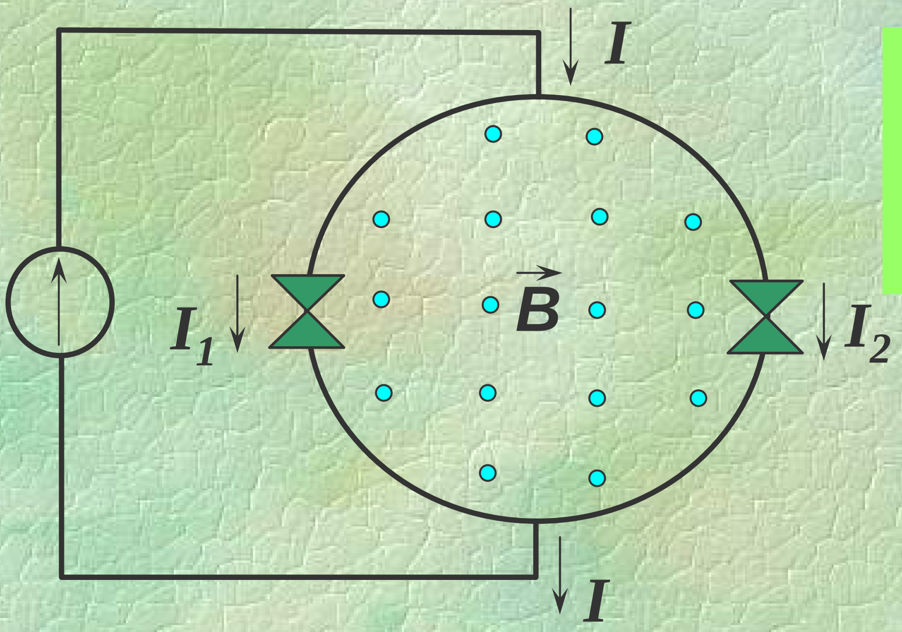


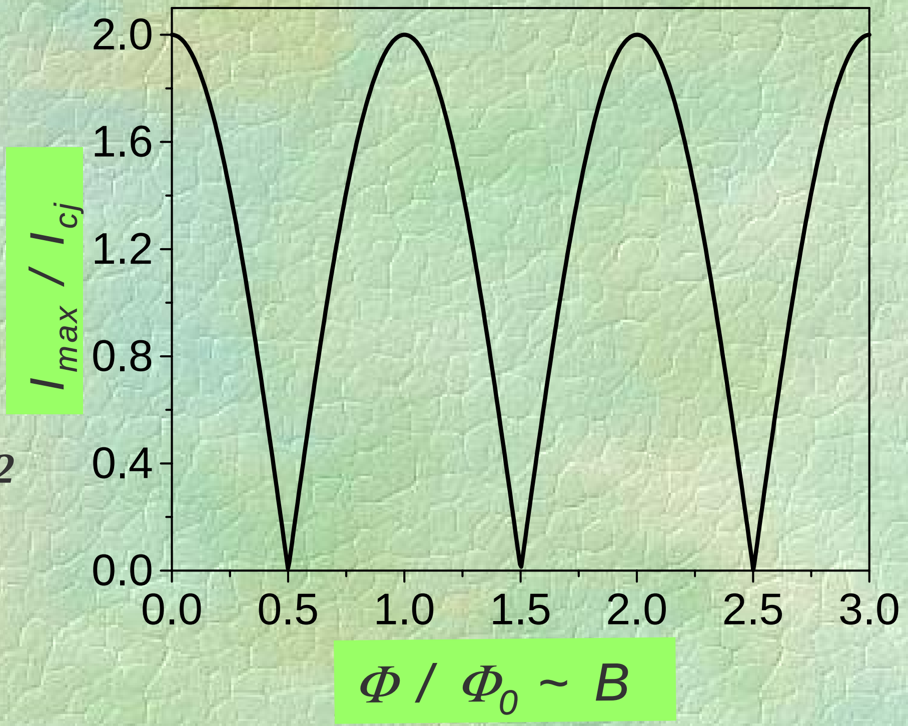
Josephson Junction	Pendulum
Phase difference, φ	Deflection, ϕ
Total current, *I*	Applied torque, *Mgr*
Josephson current, I_j	$mgL\sin\phi$
$(\hbar/2e)C$	Moment of inertia, Γ
$(\hbar/2e)(1/R)$	Viscous damping, η
Voltage, $V=(\hbar/2e)(d\varphi/dt)$	Angular velocity, $\omega = d\phi/dt$

What is new, starts here....

Two Josephson junctions in a magnetic field

-- The ideal DC SQUID

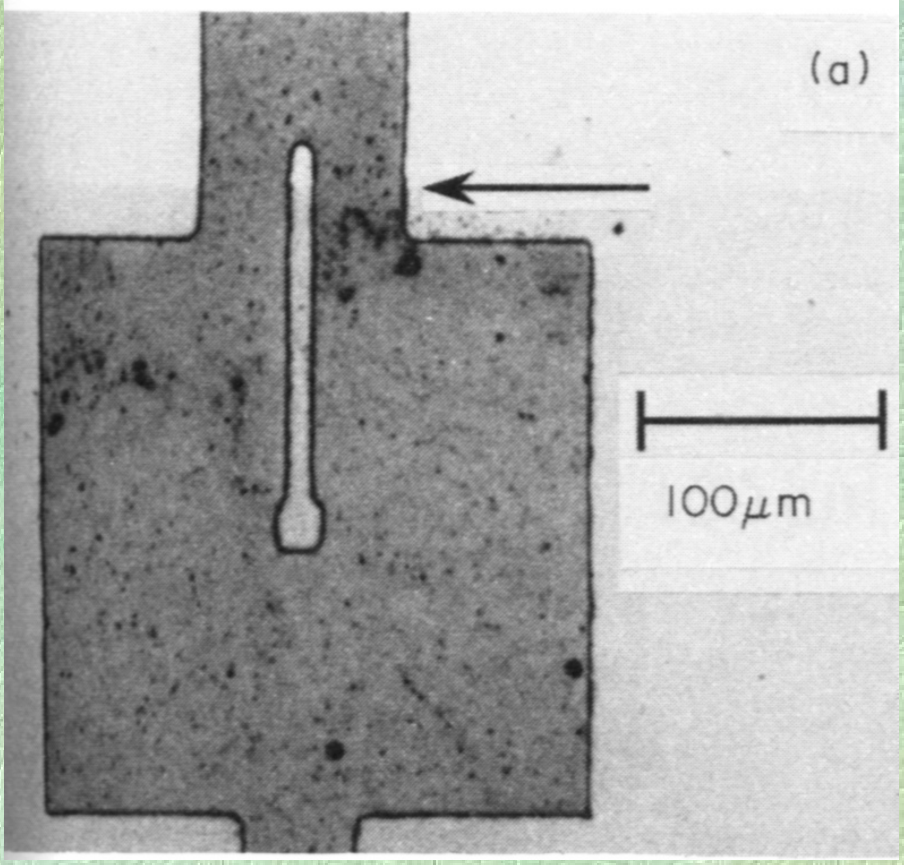


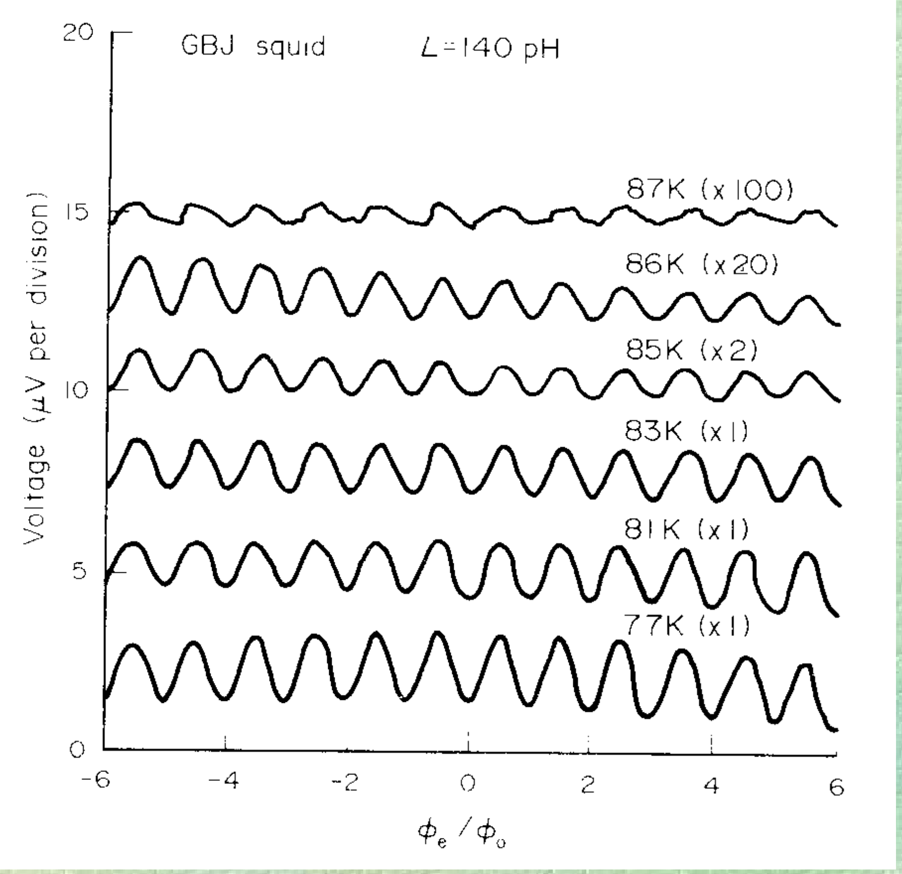


$$I_{\max} = 2I_{cj}\left|\cos\left(\frac{\pi\Phi}{\Phi_o}\right)\right|$$

Two Josephson junctions in a magnetic fields

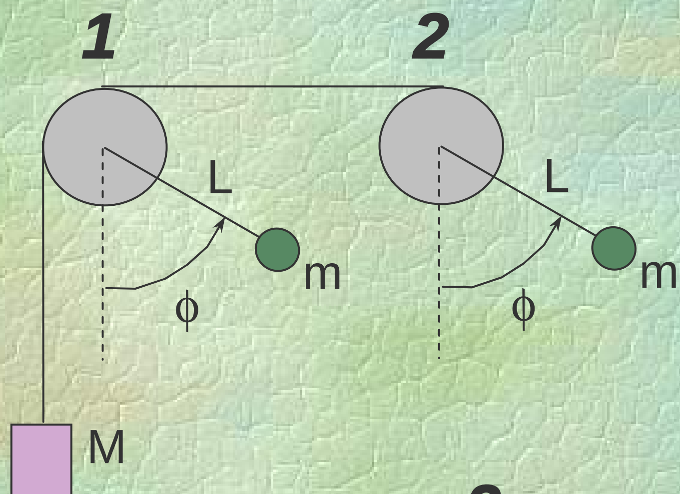
-- The real DC SQUID



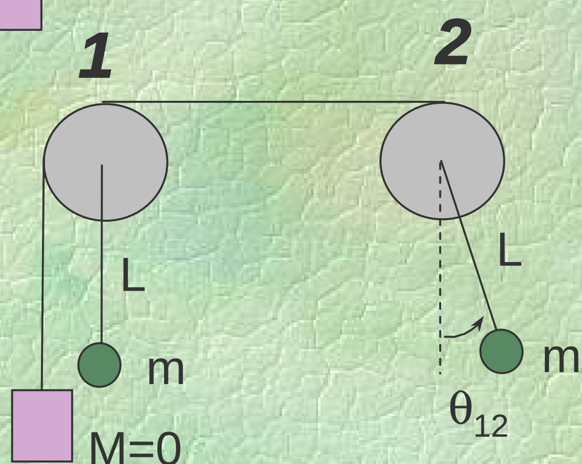


Modelling two Josephson junctions in a magnetic field: *--The DC SQUID*

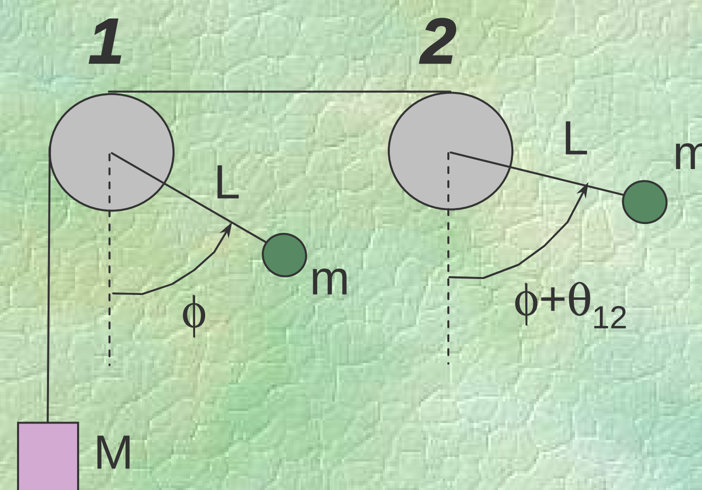
$H = 0$
$I \neq 0$



$H \neq 0$
$I = 0$



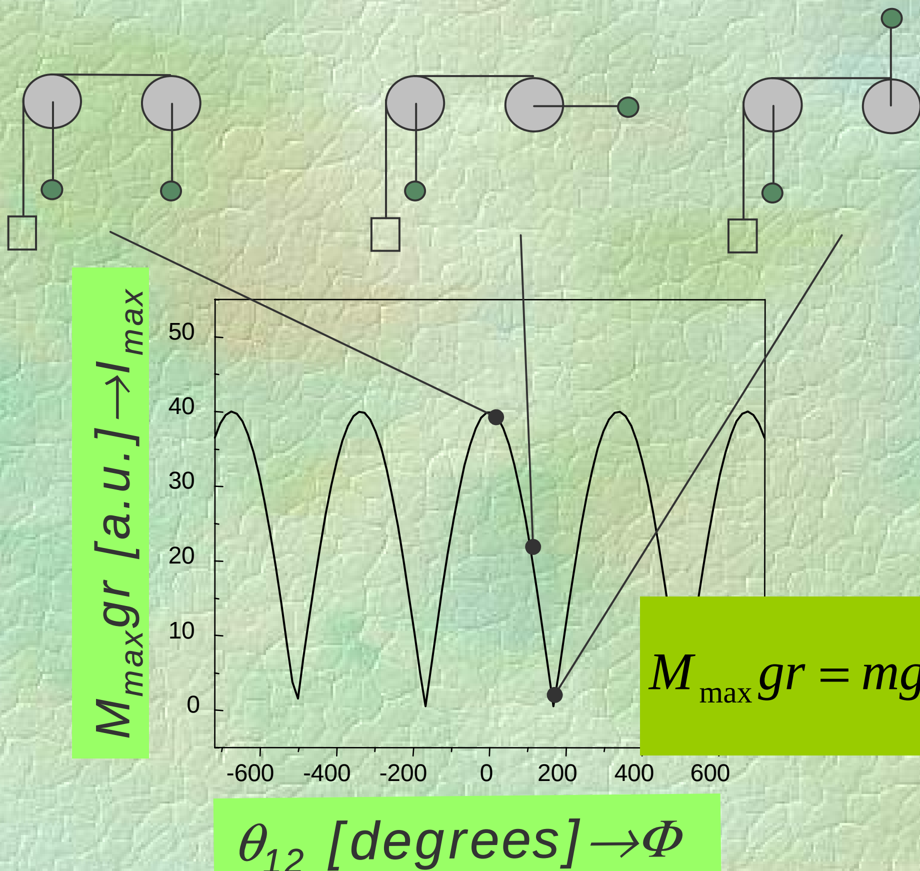
$H \neq 0$
$I \neq 0$



Modelling two Josephson junctions in a magnetic field

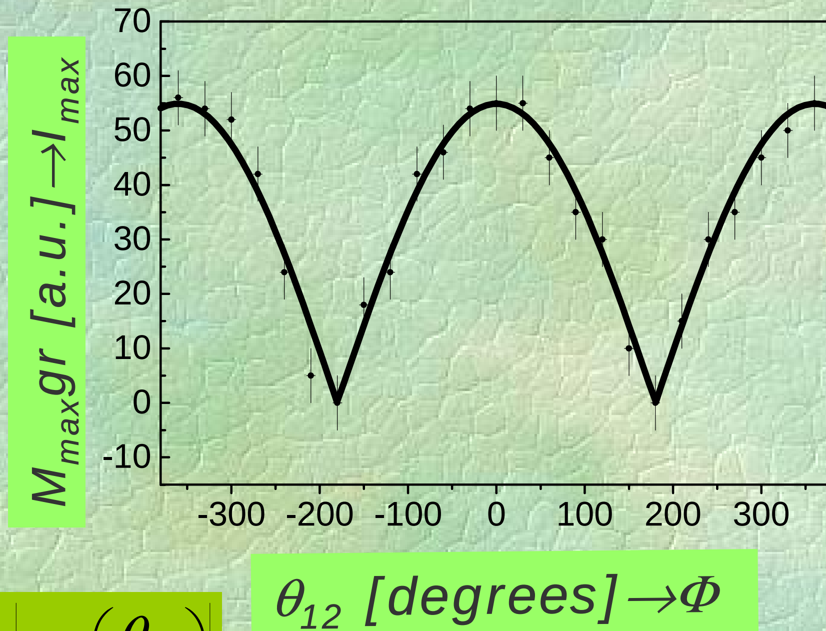
--The DC SQUID

Theoretical result

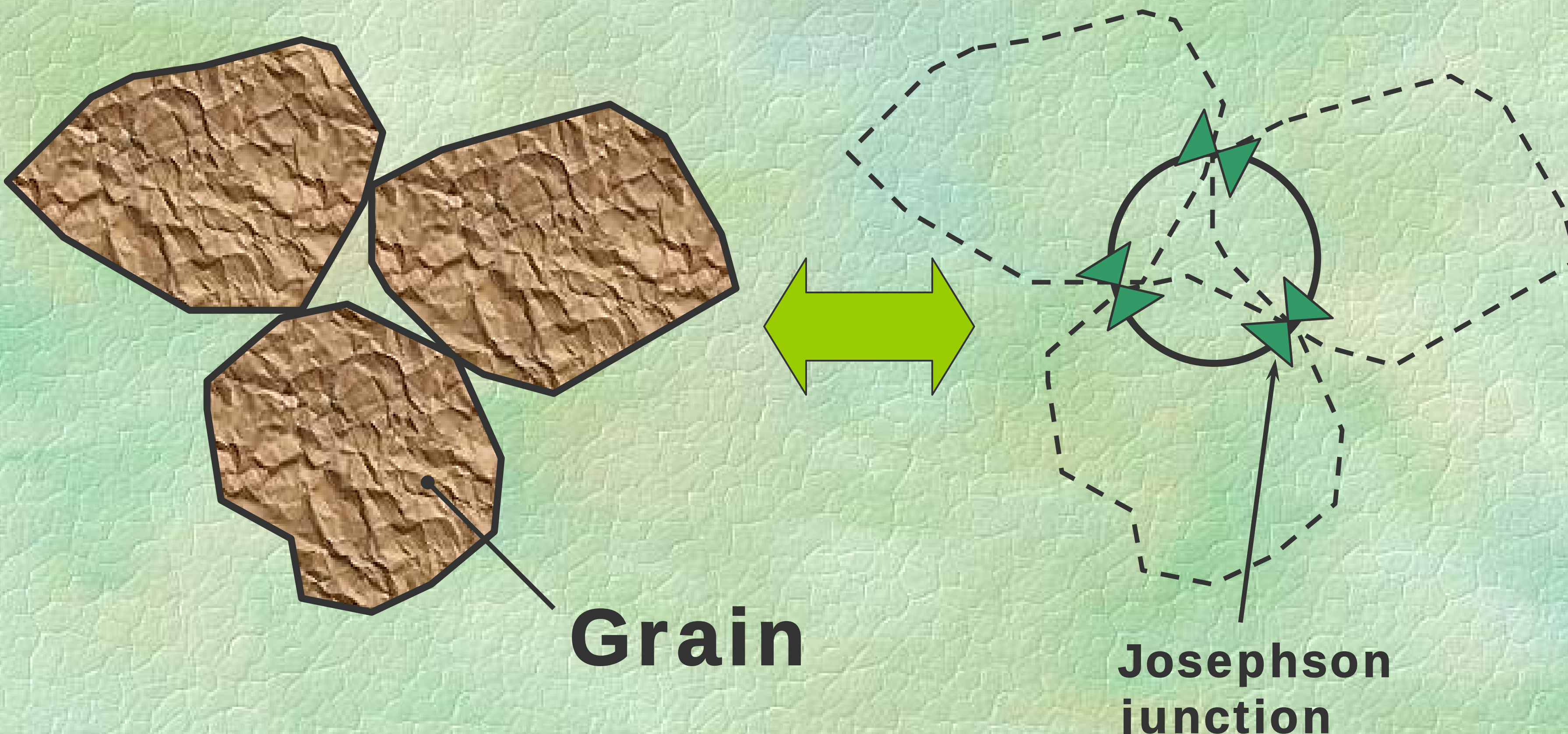


$$M_{\max} gr = mgL \left| \cos\left(\frac{\theta_{12}}{2}\right) \right|$$

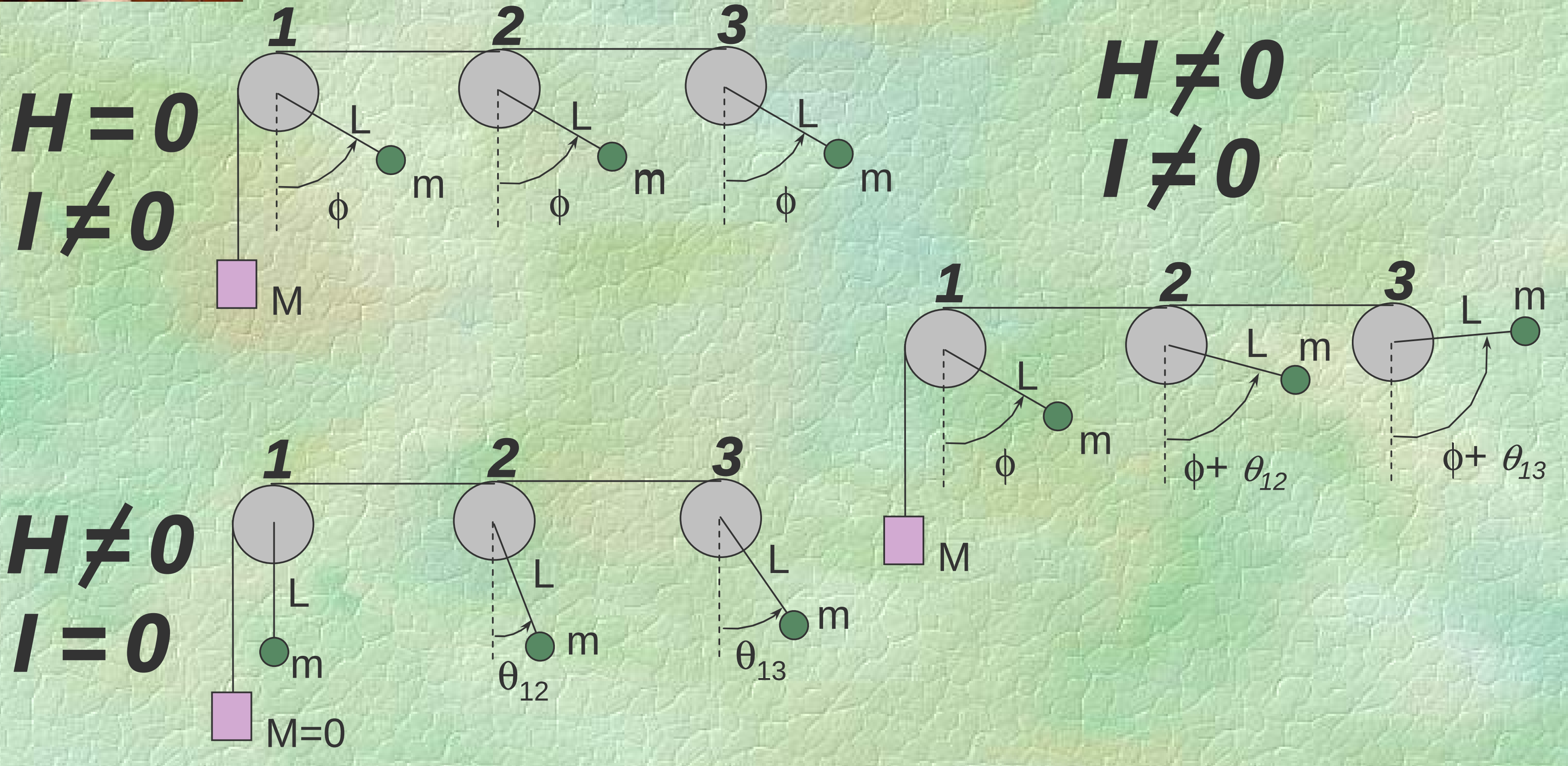
Experimental result



Three Josephson junctions -- *the ideal tricrystal*



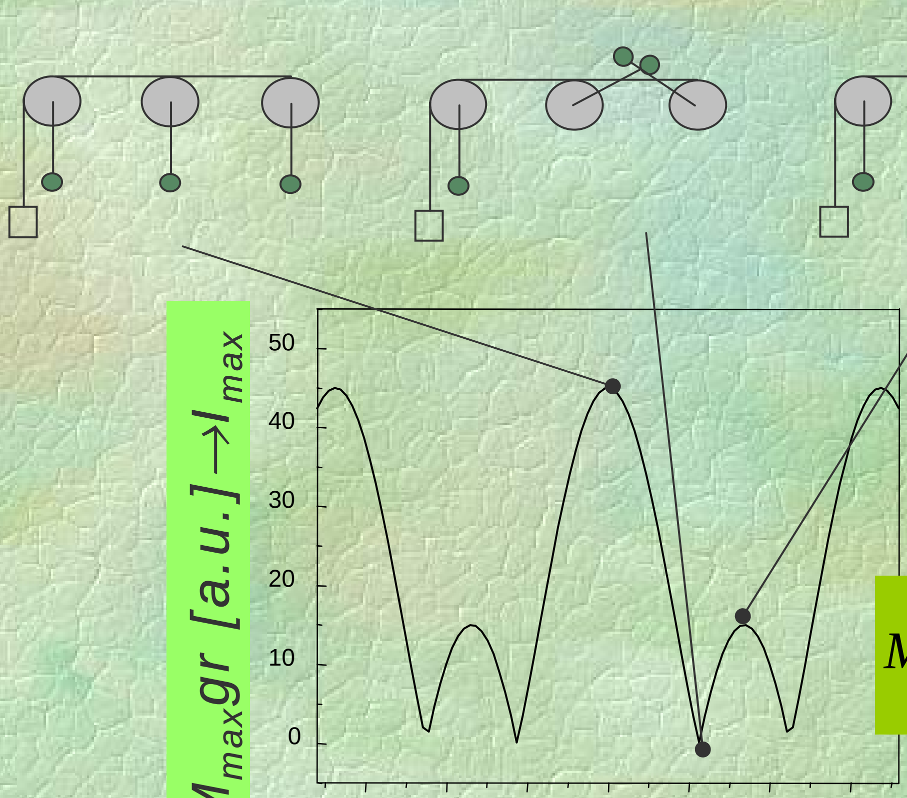
Modelling three Josephson junctions in a magnetic field: *--The tricrystal*



Modelling three Josephson junctions in a magnetic field

--The tricrystal

Theoretical result



$M_{max}gr$ [a.u.] $\rightarrow I_{max}$

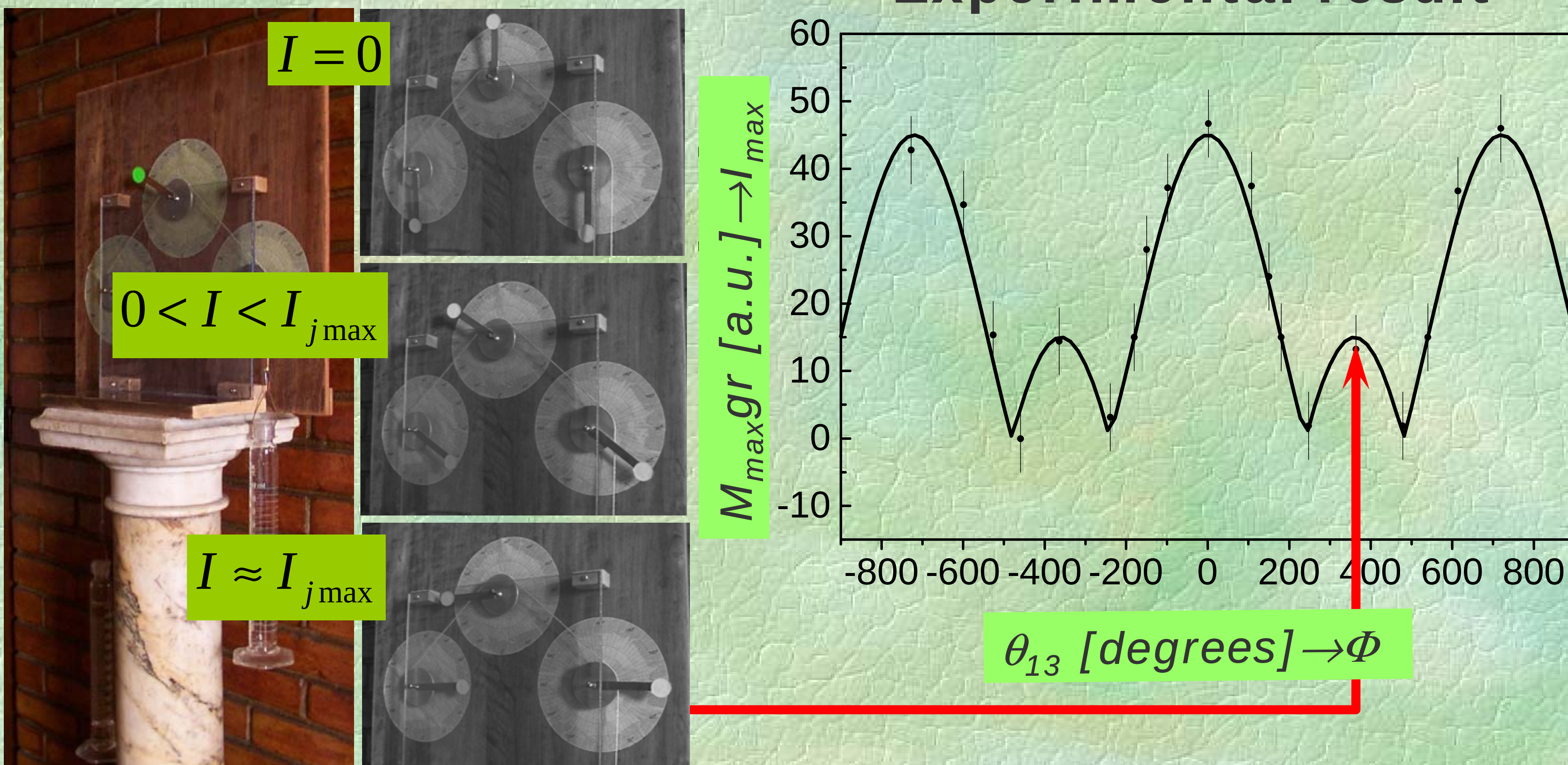
θ_{13} [degrees] $\rightarrow \Phi$

$$M_{\max} gr = mgL\left|1 + 2\cos\left(\frac{\theta_{13}}{2}\right)\right|$$

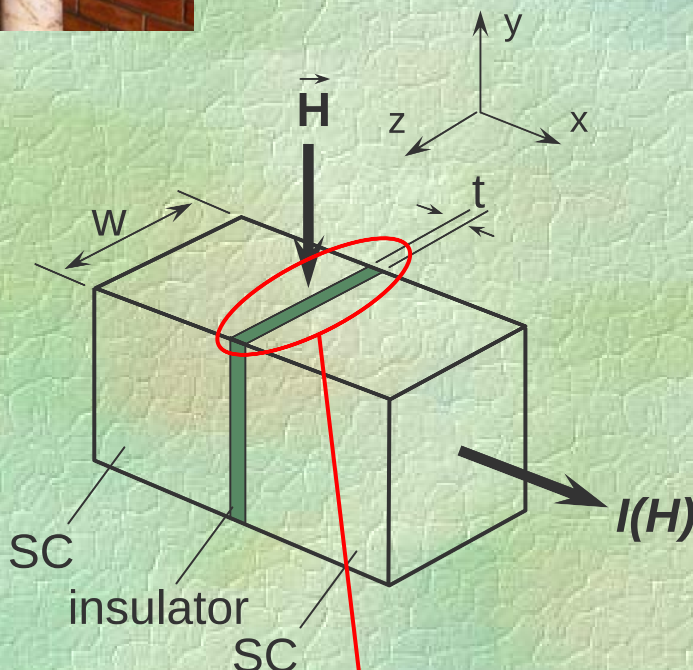
Modelling three Josephson junctions in a magnetic field

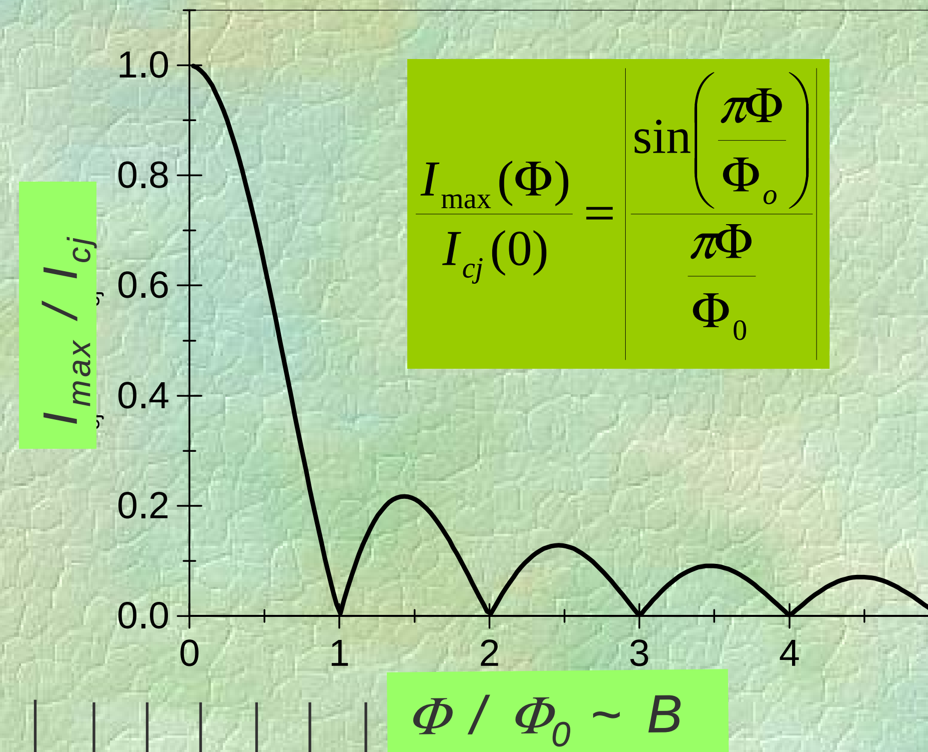
--The tricrystal

Experimental result

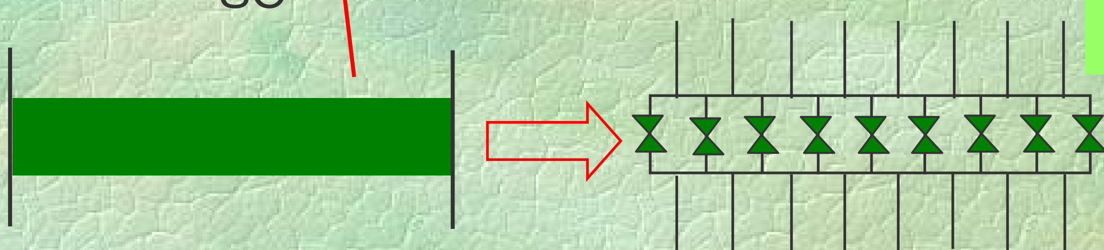


Infinite Josephson junctions in a magnetic field
-- the ideal “extended” junction



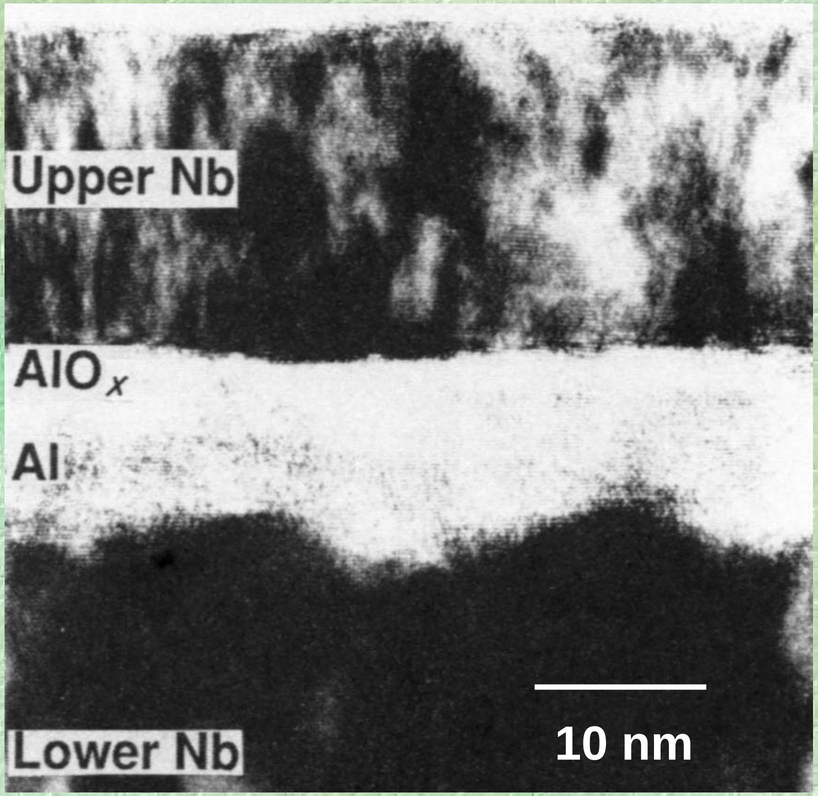


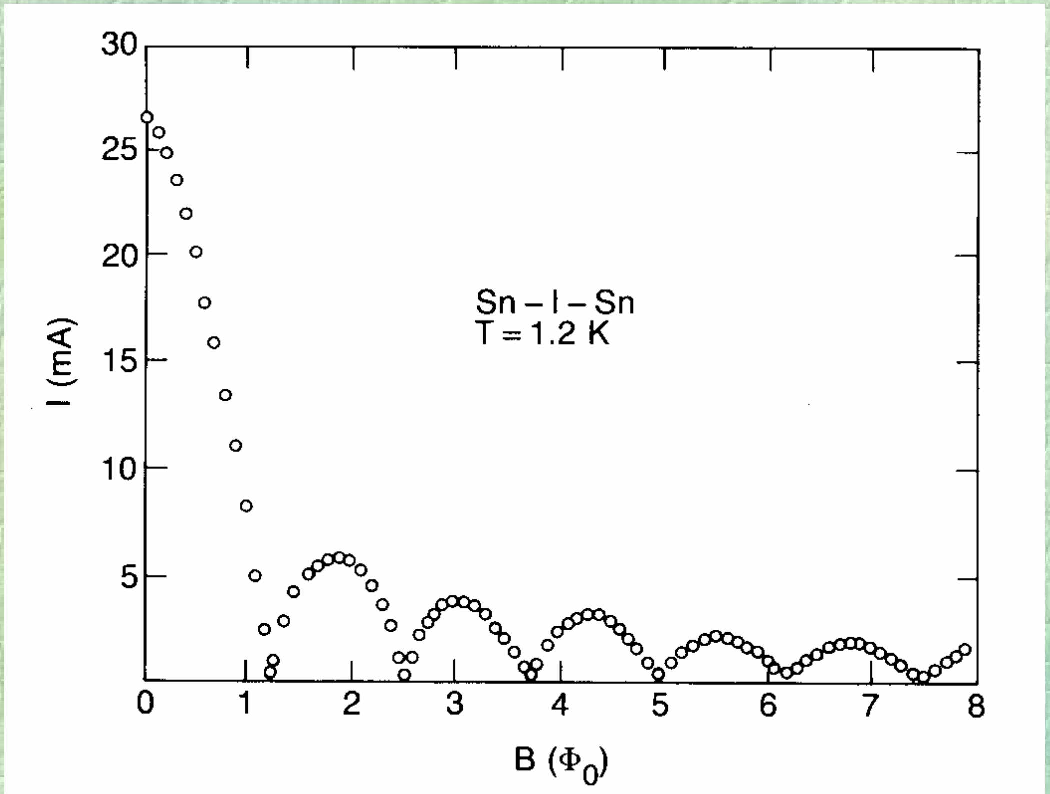
$$\frac{I_{\max}(\Phi)}{I_{cj}(0)} = \left| \frac{\sin\left(\frac{\pi\Phi}{\Phi_o}\right)}{\frac{\pi\Phi}{\Phi_0}} \right|$$



Three Josephson junctions in a magnetic field

-- the real "extended" junction



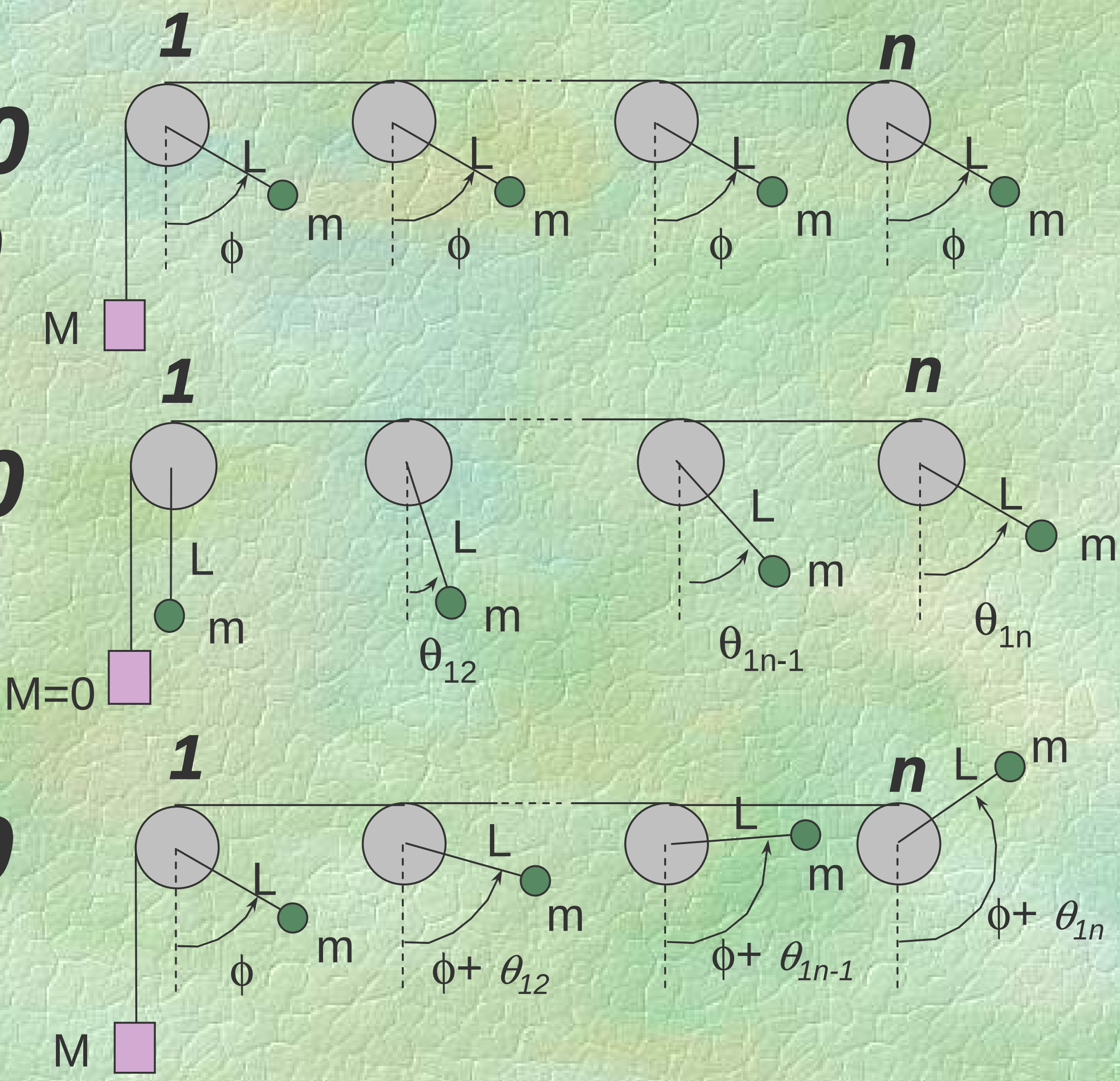


Modelling an infinite set of Josephson junctions

H = 0, I ≠ 0

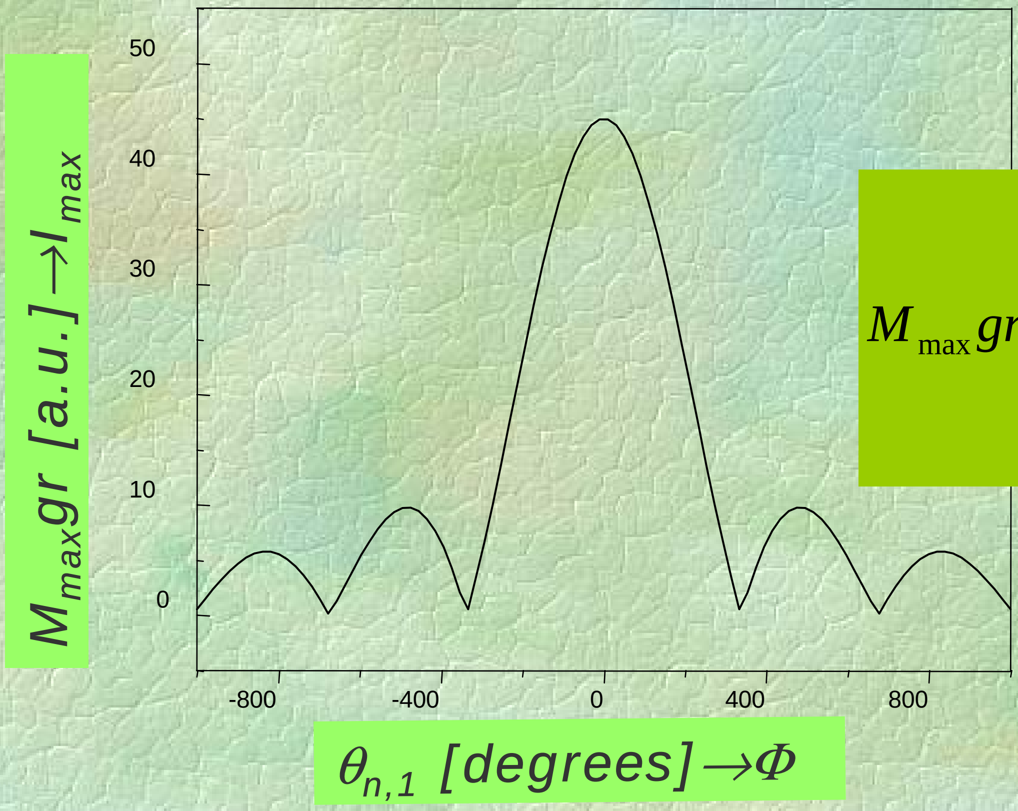
H ≠ 0, I = 0

H ≠ 0, I ≠ 0



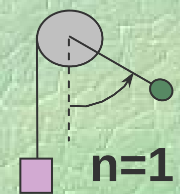
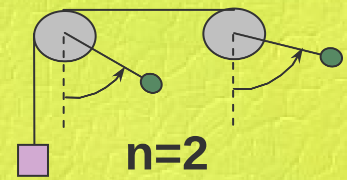
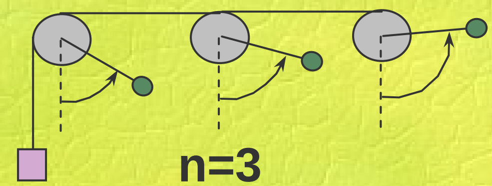
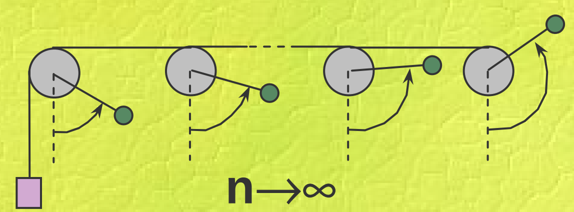
Modelling an infinite set of Josephson junctions in a magnetic field
--The “extended” junction

Theoretical result



$$M_{\max} gr = m_o gL \left| \frac{\sin\left(\frac{\theta_{n,1}}{2}\right)}{\frac{\theta_{n,1}}{2}} \right|$$

A short summary

Superconducting scenario	Pendulum analogue
Concentrated Josephson junction	 n=1
DC SQUID	 n=2
”Tricrystal”	 n=3
Extended Josephson junction	 n→∞

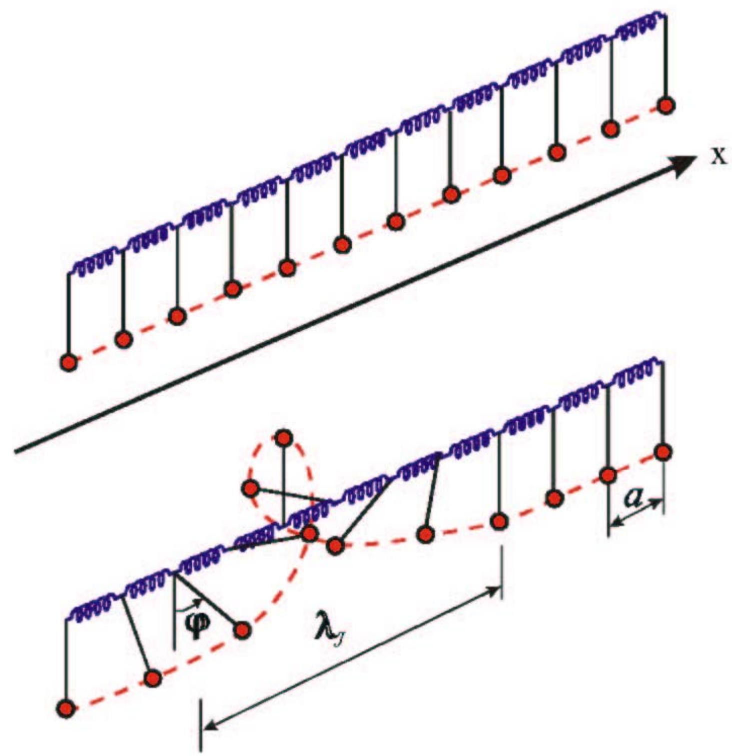
Concluding remarks

A set of rigid pendula linked by a common rope reproduces the magnetic field dependence of the Josephson maximum current of sets of concentrated Josephson junctions in parallel

The mechanical analog is easy to set up and work experimentally, and the theoretical calculations can be performed by elementary methods

Our analog lends itself to problems and projects suitable for students work

Of course there were mechanical models before!



- Detailed experimental study ($N = 25$): M.Cirillo, R.D.Parmentier, and B.Savo, Physica D **3**, 565 (1981).