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Bean–Livingston barriers in ideal type-II superconductors Hollow cylinders

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Abstract

The exact solution for circular vortices in ideal type-II superconducting cylinders is found in the presence of a circular magnetic field. The Gibbs free energy of the system for different values of the applied field is calculated and discussed in terms of surface barriers.

1. Introduction

The discussion of vortex ring nucleation and dynamics in solid superconducting cylinders has been rarely discussed in the literature. It was phenomenologically approached for example, by Tinkham [1], who considered the appearance of such vortices in the cylinder as a result of a circular magnetic field at its surface associated to a longitudinal transport current. Genenko [2], on the other hand, found the exact solution for a simple vortex ring [3] also in the case of a solid cylinder at zero applied field.

In the present work the field distribution of an ideal type-II superconducting hollow cylinder is calculated in the London approximation considering not only the presence of a circular vortex but also a circular magnetic field imposed, for example, by means of a current-carrying conducting wire coaxially arranged. The Gibbs free energy is also calculated for different values of the circular applied field and for different dimensions of the cylinder, showing the existence of Bean–Livingston type barriers [4], which are physically discussed, as well as some other peculiarities of the system related to its special symmetry.

2. Calculation of the magnetic field distribution

The behavior of the magnetic field in the mixed state of an ideal type-II superconductor may be represented by the modified second London equation:

$$\boldsymbol{h} + \lambda^2 \operatorname{curl}(\operatorname{curl} \boldsymbol{h}) = \Phi_0 \delta_2(\boldsymbol{\rho} - \boldsymbol{r}) \frac{\boldsymbol{h}}{|\boldsymbol{h}|}, \quad (1)$$

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where λ is the London penetration depth, $\boldsymbol{h}$ is the microscopic magnetic field, $\boldsymbol{r}$ is the position of the vortex axis, and Φ_0 is the flux quantum [1,5,6].

Considering our special case where the vortex is thoroidal and the applied field is circular it is easy to see that $\boldsymbol{h}$ will depend only on the radial and longitudinal coordinates of the cylinder and will have only a component in the circular direction. Then we may transform our vector equation (1) into a more tractable scalar one:

$$\frac{\partial^2 h}{\partial \rho^2} + \frac{\partial h}{\rho \partial \rho} - \left[\frac{1}{\rho^2} + \frac{1}{\lambda^2}\right] h + \frac{\partial^2 h}{\partial z^2} = -\frac{\Phi_0}{\lambda^2} \delta(\rho - r) \delta(z), \tag{2}$$

with the boundary conditions: $h(a, z) = H_0$ and $h(b, z) = H_0(a/b)$, where a and b are the radii of the inner and outer surfaces of the cylinder, respectively, H_0 is the applied field at the inner wall of the cylinder, and r is the radius of the thoroidal vortex.

To solve Eq. (2) it is convenient to consider the superposition principle [7], by assuming $\boldsymbol{h} = \boldsymbol{h}_1 + \boldsymbol{h}_2$, where $\boldsymbol{h}_2$ is the field distribution into the sample in the Meissner state and $\boldsymbol{h}_1$ is the field related to the vortex presence in the material. Then after substituting in Eq. (2), we obtain two different mathematical problems:

$$\frac{\partial^2 h_1}{\partial \rho^2} + \frac{\partial h_1}{\rho \partial \rho} - \left[\frac{1}{\rho^2} + \frac{1}{\lambda^2}\right] h_1 + \frac{\partial^2 h_1}{\partial z^2} = -\frac{\Phi_0}{\lambda^2} \delta(\rho - r) \delta(z), \tag{3}$$

with the boundary conditions $\boldsymbol{h}_1(b, z) = 0$, $\boldsymbol{h}_1(a, z) = 0$ and $\boldsymbol{h}_1(\rho, \infty) = 0$ (as pointed out by Genenko [2] if we keep in mind that an ideal solenoid does not create any field in outer space), and

$$\frac{\partial^2 h_2}{\partial \rho^2} + \frac{\partial h_2}{\rho \partial \rho} - \left[\frac{1}{\rho^2} + \frac{1}{\lambda^2}\right] h_2 = 0, \tag{4}$$

with the boundary conditions $\boldsymbol{h}_2(a) = H_0$ and $\boldsymbol{h}_2(b) = H_0(a/b)$.

To solve Eq. (3) we made use of the method of separation of variables [8] and obtained a solution as a series of Bessel and Neumann functions as follows:

$$h_1(\rho, z) = \frac{\pi^2}{2} \frac{\Phi_0}{\lambda^2} r \sum_n \frac{\mu_n^2}{\sqrt{\mu_n^2 + 1/\lambda^2}} \exp\left(-\sqrt{\mu_n^2 + 1/\lambda^2}\,|z|\right) f_{11}(r, a) f_{11}(\rho, a), \tag{5}$$

where the functions $f_{11}(r, a)$ and $f_{11}(\rho, a)$ are given by

$$f_{\alpha\beta}(x, y) = J_\alpha(\mu_n x) N_\beta(\mu_n y) - J_\beta(\mu_n y) N_\alpha(\mu_n x), \tag{6}$$

and where μ_n are the positive solutions of the equation

$$J_1(\mu_n b) N_1(\mu_n a) - J_1(\mu_n a) N_1(\mu_n b) = 0. \tag{7}$$

The solution of Eq. (4) can be obtained in a simpler way since it can be easily transformed to the equation of imaginary-argument Bessel functions:

$$h_2(\rho) = H_0 \frac{\left[\frac{a}{b} K_1(a/\lambda) - K_1(b/\lambda)\right] I_1(\rho/\lambda) + \left[I_1(b/\lambda) - \frac{a}{b} I_1(a/\lambda)\right] K_1(\rho/\lambda)}{K_1(a/\lambda) I_1(b/\lambda) - K_1(b/\lambda) I_1(a/\lambda)}, \tag{8}$$

where I_1 and K_1 are the Infeld and McDonald functions.

3. Calculation of the Gibbs free energy

The Gibbs free energy of the system is given by the integral [5]

$$G=\int \mathrm{d}v\left[\frac{\boldsymbol{h}^2+\lambda^2(\text{curl } \boldsymbol{h})^2}{8\pi}-\frac{\boldsymbol{H}\cdot\boldsymbol{h}}{4\pi}\right], \tag{9}$$

where $\boldsymbol{H}$ is the magnetic field generated by the coaxial wire in the absence of the superconductor and the integral is taken over the whole superconducting region.

Using the vector property [9]:

$$(\text{curl } \boldsymbol{h})^2=\boldsymbol{h}\cdot\text{curl}(\text{curl } \boldsymbol{h})+\text{div}(\boldsymbol{h}\times\text{curl } \boldsymbol{h}) \tag{10}$$

we transform Eq. (8) in

$$G=\frac{1}{8\pi}\int \mathrm{d}v\boldsymbol{h}\cdot\left[\boldsymbol{h}+\lambda^2\text{ curl}(\text{curl } \boldsymbol{h})\right]+\frac{\lambda^2}{8\pi}\int \mathrm{d}v[\text{div}(\boldsymbol{h}\times\text{curl } \boldsymbol{h})]-\frac{1}{4\pi}\int \mathrm{d}v\boldsymbol{H}\cdot\boldsymbol{h}. \tag{11}$$

The expression in brackets in the first term of the right side of Eq. (11) can be substituted, following Eqs. (1) and (2), by $\Phi_0\delta(\rho-r)\delta(z)$. Then in our special case

$$\int \mathrm{d}v\boldsymbol{h}\cdot\left[\boldsymbol{h}+\lambda^2\text{ curl}(\text{curl } \boldsymbol{h})\right]=\int \mathrm{d}vh\Phi_0\delta(\rho-r)\delta(z)=2\pi r\Phi_0\left[h_1(r-\xi)+h_2(r-\xi)\right], \tag{12}$$

where the evaluation takes place at $r-\xi$ instead of r, since the field actually saturates at the axis of the vortex within the scale of ξ, the coherence length, as was suggested by Genenko [2].

The second term can be transformed by using Stokes theorem [9] into a surface integral as follows:

$$\int \mathrm{d}v[\text{div}(\boldsymbol{h}\times\text{curl } \boldsymbol{h})]=\oint \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{h}\times\text{curl } \boldsymbol{h}=\oint \mathrm{d}\boldsymbol{s}\cdot(\boldsymbol{h}_1+\boldsymbol{h}_2)\times\text{curl}(\boldsymbol{h}_1+\boldsymbol{h}_2), \tag{13}$$

where the integration is taken over the sample and core surfaces.

All the previous integrals related with $\boldsymbol{h}_1$ are zero over the sample surfaces because the vortex field at the surfaces is zero. Then we have

$$\oint \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{h}\times\text{curl } \boldsymbol{h}=\oint_s \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{h}_2\times\text{curl } \boldsymbol{h}_2+\oint_{\text{core}} \mathrm{d}\boldsymbol{s}\cdot\boldsymbol{h}\times\text{curl } \boldsymbol{h}, \tag{14}$$

where the first term in the right side of Eq. (14) does not depend on the vortex radius while the second will be zero for a core of small dimensions.

The last term in the right side of Eq. (11) can be written as

$$\frac{1}{4\pi}\int \mathrm{d}v\boldsymbol{H}\cdot\boldsymbol{h}=\frac{1}{4\pi}\int \mathrm{d}v\boldsymbol{H}\cdot\boldsymbol{h}_1+\frac{1}{4\pi}\int \mathrm{d}v\boldsymbol{H}\cdot\boldsymbol{h}_2. \tag{15}$$

The field $\boldsymbol{h}_2$ does not depend on the vortex radius, so the first term in the right side of Eq. (14) does not involve physical information relevant to our problem. Finally the calculation of the first integral in the right side of Eq.

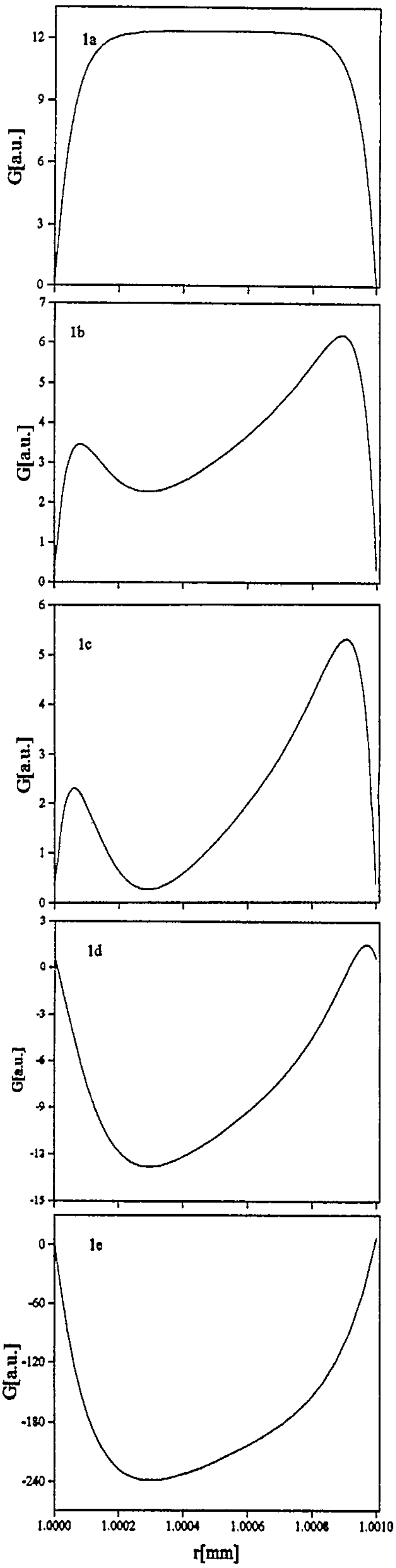


Fig. 1. Profiles of the Gibbs free energy from $r=a$ to $r=b=a+10\lambda$ for different values of the field H_0 at the inner surface of the hollow cylinder: $H_0=0$ (a), $H_0=0.8H_{c1}^{HC}$ (b), $H_0=H_{c1}^{HC}$ (c), $H_0=2H_{c1}^{HC}$ (d) and $H_0=20H_{c1}^{HC}$ (e)

(15) is straightforward. Hence, if we substitute Eqs. (5) and (8) in Eqs. (12) and (15) and solve, and finally insert those results as well as Eq. (14) in Eq. (11), we obtain

$$G=\frac{1}{4\pi}\left(\frac{\pi r}{\lambda}\right)^2\Phi_0^2\sum_n\frac{\mu_n^2}{\sqrt{\mu_n^2+1/\lambda^2}}\frac{J_1^2(\mu_n b)}{J_1^2(\mu_n b)-J_1^2(\mu_n a)}f_{11}(r,a)f_{11}(r-\xi,a)$$
$$+\frac{r}{4}\Phi_0 H_0\frac{\left[\frac{a}{b}K_1(a/\lambda)-K_1(b/\lambda)\right]I_1(r/\lambda)+\left[I_1(b/\lambda)-\frac{a}{b}I_1(a/\lambda)\right]K_1(r/\lambda)}{K_1(a/\lambda)I_1(b/\lambda)-K_1(b/\lambda)I_1(a/\lambda)}$$
$$-\frac{\pi^3}{\lambda^2}ra\Phi_0 H_0\sum_n\frac{\mu_n}{\mu_n^2+1/\lambda^2}\frac{J_1^2(\mu_n b)}{J_1^2(\mu_n b)-J_1^2(\mu_n a)}f_{11}(r,a)\left[f_{01}(a,b)-f_{10}(a,a)\right]. \quad (16)$$

The first term in Eq. (16) is related (as pointed out by Genenko [2] for the case of zero applied field), to the interaction of the vortex with the inner and outer surfaces of the cylinder and it also contains the natural tendency of the vortex to contract in a way to minimize its energy. The second term is the energy associated to the interaction of the external field with the vortex ring which "pushes" it into the superconducting region. The third term has not a straightforward interpretation and is related to the interaction between the vortex and the field generated by the current flowing through the wire.

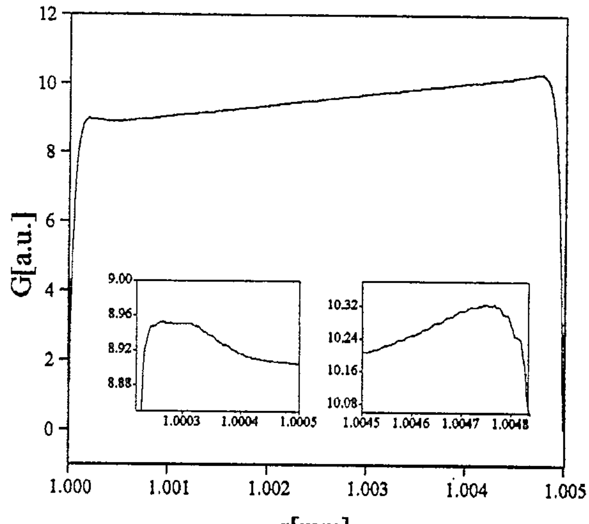
4. Results and discussion

Figs. 1(a–e) represent the Gibbs free energy dependence of the system on the vortex radius for different values of the external field considering a thin-walled hollow cylinder ($b-a=10\lambda$). As in the rest of the calculations here presented, a simplificated version of the expression of Eq. (16) was evaluated (see Appendix A).

In order to compare the results for different applied fields in an illustrative way, we defined the "first critical field" of the hollow cylinder, H_{c1}^{HC} as the field required to obtain the same values of G at $r=a$ and at the minimum located between a and b. It must be stressed that this is only a practical definition, since the first critical field in our case cannot be defined as simply as it can be done for a semi-infinite superconductor limited by a plane surface, as is usually reported [1,4]. Our definition is illustrated in Fig. 1(c), where $H_0=H_{c1}^{HC}$. While Figs. 1(a) and 1(b) (where $H<H_{c1}^{HC}$) show an energy slope contrary to the entrance of vortices from both surfaces, Fig. 1(c) (where $H_0=H_{c1}^{HC}$) clearly displays barriers of the type first reported by Bean and Livingston for semi-infinite superconductors in the presence of a magnetic field [4]. The asymmetry of the two barriers should be pointed out, which may be expected from the cylindrical symmetry of the problem. Fig. 1(d) ($H_0>H_{c1}^{HC}$) shows the curious case in which the inner barrier has been "destructed" (i.e., the vortices can penetrate freely from the interior), whilst the outer one still keeps its maximum. Finally, Fig. 1(c) (where $H_0\gg H_{c1}^{HC}$) displays a situation of free entrance of vortices from both sides of the cylinder.

In Fig. 2 it is represented the Gibbs free energy of the system for $b\gg a$. As can be seen from the figure the surface barriers are meaningful only very near the frontier and can be neglected if we analyze the bulk superconducting properties. Actually, it can be concluded from Fig. 2 that the bulk tendency of the vortex is to "shrink" to the inner radius of the cylinder, as was pointed out earlier by other authors [1,6] based on qualitative arguments.

As remarked in Section 3, the interpretation of the third term of the right side of Eq. (16) is not straightforward. For getting a better understanding of this, we plotted the contribution of the first two terms of the right side of Eq. (16) in Fig. 3, and the contribution of all the terms (the third one included) in Fig. 4. Then, Fig. 3 contains the effect of three physical phenomena: the external field "pushing" the vortex from the



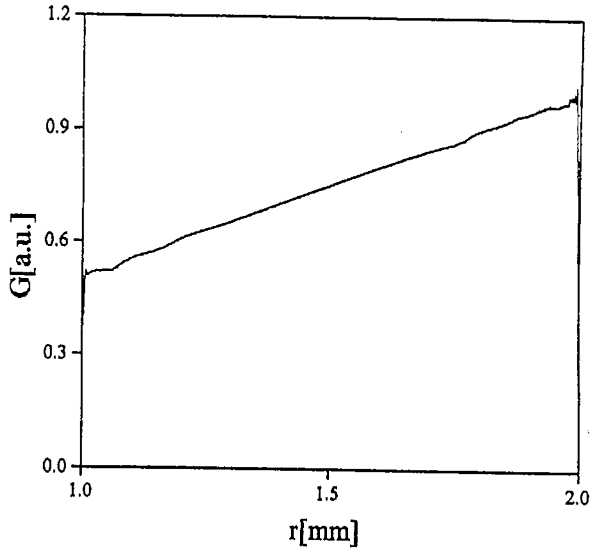


Fig. 2. Gibbs free energy profiles from $r = a$ to $r = b = a + 50\lambda$ corresponding to some $H_0 > 0$. The left and right inserts shows close-ups of the profile from $r = a$ to $r = a + 5\lambda$, and from $r = b - 5\lambda$ to $r = b$, respectively.

Fig. 3. Gibbs free energy profile for a very thick walled cylinder corresponding to some $H_0 > 0$ without taking into account the last term in the right side of Equation (16) (see text).

surfaces, the attraction of the vortex to the surfaces, and the tendency of the vortex to "shrink" in order to decrease its free energy (which is independent of the applied field). The first two effects are responsible for the Bean-Livingston type of barriers, while the third one justifies the bulk linear slope of the free-energy profile.

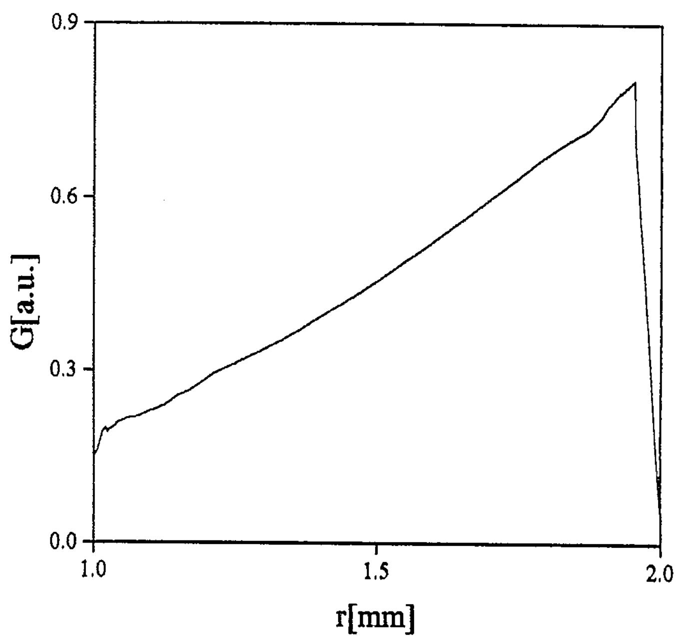


Fig. 4. Gibbs free energy profile similar to the one of Figure 3, but considering the whole expression (16).

Fig. 4 shows that the basic effect of the addition of the last term in the right side of the expression (16) is to tilt further this bulk slope, and to separate it from the linear behavior. Then, the interaction of the vortex with the applied field increases its tendency to shrink to the inner surface of the cylinder.

5. Conclusions

Based on the London approximation, we calculated the microscopic field distribution in a hollow superconducting cylinder with a thoroidal vortex under the effect of a circular magnetic field.

The calculation of the Gibbs free energy for different radii of the vortex revealed the existence of surface barriers analogous to the ones reported by Bean and Livingston for the case of a semi-infinite superconductor limited by a plane surface. In our case, however, the barriers associated to both surfaces of the hollow cylinder are not symmetric.

The effect of the surface barriers is important only close to the inner and outer radii of the cylinder, while in the bulk, a dominant tendency of the circular vortex to shrink down to the inner surface was observed.

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Appendix A

The expression (16) represents an exact solution of the problem, but the computation of the numeric values was found to be problematic. To reduce the computer time of calculation we made use, considering the slow convergence of the series, of the asymptotic expression of Bessel functions [10]:

$$J_1(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3\pi}{4}\right), \tag{A.1}$$

$$N_1(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{3\pi}{4}\right), \tag{A.2}$$

$$I_1(z) = \sqrt{\frac{1}{2\pi z}}\ \mathrm{e}^{2}, \tag{A.3}$$

$$K_1(z) = \sqrt{\frac{\pi}{2z}}\ \mathrm{e}^{-2}. \tag{A.4}$$

Substituting Eqs. (A.1–A.4) in Eq. (16) and solving, we obtain

$$G = \frac{\Phi_0^2}{2\lambda^2} \frac{r}{b-a} \sum_n \frac{\cos(\mu_n \xi) - \cos(2\mu_n(a-r))}{\sqrt{\mu_n^2 + 1/\lambda^2}}$$
$$+ \Phi_0 H \frac{\sqrt{rab}}{4} \frac{(b/a)\sqrt{1/a}\sinh((r-a)/\lambda) - \sqrt{1/b}\sinh((r-b)/\lambda)}{\sinh((b-a)/\lambda)}$$
$$- 4\pi \frac{\Phi_0 H_0}{\lambda^2} \frac{a}{b-a} \sqrt{\frac{r}{a}} \sum_n \frac{\sin(\mu_n(a-r))}{\mu_n(\mu_n^2 + 1/\lambda^2)} \left(\sqrt{\frac{a}{b}}(-1)^n - 1\right), \tag{A.5}$$

which is also an infinite series but numerically more tractable. Here $\mu_n = n\pi/(b-a)$, is the asymptotic expansion of the zeros given by Eq. (6) [10].

References

[1] M. Tinkham, Introduction to Superconductivity (McGraw-Hill, New York, 1975).
[2] Yu.A. Genenko, Phys. Rev. B 49 (1994) 6950.
[3] G. Carneiro, Physica C 183 (1991) 360.
[4] C.P. Bean and J.D. Livingston, Phys. Rev. Lett. 12 (1963) 14.
[5] R.P. Huebener, Magnetic Flux Structures in Superconductors (Springer, Berlin, 1979).
[6] D.R. Tilley and J. Tilley, Superfuidity and Superconductivity (John Wiley, New York, 1974).
[7] J.D. Jackson, Classical Electrodynamics (John Wiley, New York, 1965).
[8] A.N. Tikhonov and A.A. Samarskii, Equations of Mathematical Physics (Mir, Moscow, 1972).
[9] M.R. Spiegel, Vector Analysis (Schaum Publishing Company, New York, 1959).
[10] Eds. M. Abramovitz and I. Stegun, Handbook of Mathematical Functions (Appl. Math. Ser. No. U.S. GPO. Washington, D.C., 1965).