Josephson junctions in a magnetic field: Insights from coupled pendula

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Josephson effects are macroscopic quantum phenomena that can be understood at the undergraduate level with the help of mechanical analogs. Although Josephson junctions in zero magnetic field can be modeled by pendulum analogs, a simple mechanical model of Josephson junctions in nonzero fields has been elusive. We demonstrate how the magnetic field dependence of the maximum Josephson current can be visualized by the analogs of a set of interconnected pendula attached to pulleys.
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I. INTRODUCTION

A Josephson junction can be defined as a superconductor–insulator–superconductor sandwich that allows superconducting tunneling, that is, the resistanceless flow of Cooper pairs through the junction. Let us assume that the dimensions of the sandwich perpendicular to the tunneling current are negligible. When a direct current is forced through such a junction,[1] the gauge invariant phase difference between the electrodes, φ, adapts to the applied current according to the Josephson equation[2,3]

$$I_j = I_{cj} \sin \varphi, \tag{1}$$

where I_j is the current in the junction and I_{cj} is its critical current. This tunneling is not dissipative unless the applied current surpasses I_{cj}.

When $I_j > I_{cj}$, the junction enters a dissipative regime where quasiparticles are allowed to flow. To account for this possibility, the electrical equivalent of the junction must include a resistor in parallel. Finally, the dimensions of many real junctions are not negligible, and a parallel capacitor must also be added, as shown in the left panel of Fig. 1. Thus the electrical equivalent of a realistic Josephson junction consists of an ideal superconductor–insulator–superconductor sandwich obeying Eq. (1) in parallel with a capacitor, C, and a resistor, R. This combination can be described by the differential equation:

$$I = \frac{\hbar}{2e} C \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e}\frac{1}{R}\frac{d\varphi}{dt} + I_{cj} \sin \varphi, \tag{2}$$

where I is the total current in the circuit consisting of a Josephson junction, a capacitor, and a resistor in parallel.

Equation (2) is analogous to the equation obeyed by a physical pendulum attached to a pulley, as shown in the right panel of Fig. 1. In the following we will refer to this model of a Josephson junction as a "pendulum." The equation of motion of the pendulum is

$$Mgr = \Gamma \frac{d^2\phi}{dt^2} + \eta \frac{d\phi}{dt} + mgL \sin \phi, \tag{3}$$

where M, g, r, Γ, ϕ, η, m, and L represent the mass hanging from the rope, the acceleration of gravity, the radius of the pulley, the moment of inertia of the pendulum, the angle between the pendulum rod and the vertical, the viscous damping, the mass of the pendulum's bob, and the length of the pendulum, respectively. By comparing Eqs. (2) and (3), we can establish a parallel between analogous magnitudes which we present in Table I.

The dynamics of the system can be described as follows. As M increases, the deflection of the pendulum increases, achieving an equilibrium state for each value of M. This equilibrium deflection increases until the position of the pendulum is horizontal, which occurs at a "critical" mass $M_{\max} = mL/r$. If the mass is further increased, the pendulum starts to swing, with an average frequency that increases with M. (Note that the angular frequency associated with the circular motion is smaller along the upper part of the path and bigger along the lower part.) The analogy to the Josephson junction is the following. As the current is increased, the Josephson phase difference across the junction accommodates to allow a nondissipative current flow. This accommodation happens until the system reaches I_{cj}, where an oscillating voltage drop appears through the junction, whose time average increases as the applied current does.

This beautiful analogy has been commonly used since the 1960s to illustrate the dynamics of a single Josephson junction in zero magnetic field,[4] and is still used when new Josephson phenomena are discovered.[5] However, when two or more junctions are involved and an external magnetic field is applied, the complexity of the existing mechanical analogy increases drastically: the magnetic field "globally" affects the system by modulating the *difference* between the φ's across the individual junctions. The situation is even more complicated if the lateral dimensions of a junction are relatively large (a "rectangular" junction), in which case it must be modeled by an infinite array of parallel junctions such as the ones described above.[2,3,6]

We propose an extension of the pendulum analog for Josephson junctions in parallel, subject to external magnetic fields. We concentrate on the pendulum analog of the field dependence of the maximum Josephson current that a junction (or a set of junctions) can bear without dissipation and demonstrate that it can be easily found experimentally. Our experience indicates that the extended model can be an important resource for a presentation of Josephson phenomena at the undergraduate level and helps understanding at higher levels.

II. COUPLED PENDULA ANALOGS FOR JOSEPHSON JUNCTIONS IN MAGNETIC FIELDS

Figures 2–4 illustrate our mechanical analog for different numbers of Josephson junctions in parallel. The idea is to couple as many rigid pendula as the number of junctions

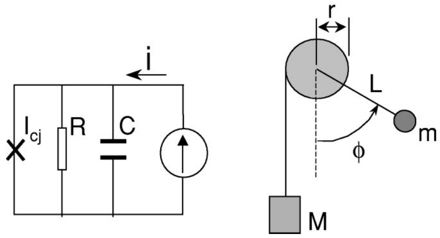

Fig. 1. Equivalent circuit for a single Josephson junction (left) and its pendulum analog (right).

under consideration by the same rope used to hang the driving mass, M. Figures 2 and 4 correspond to a dc superconducting quantum interference device (SQUID) and to a rectangular Josephson junction, respectively. Both cases are widely discussed in textbooks, because the first constitutes the most sensitive magnetometer known, and the second is the junction geometry most commonly found in practice.[2,3] Figure 3 represents three Josephson junctions that we will use as examples. In contrast to the SQUID and the rectangular junction, the case of three junctions in parallel is not widely used in practical devices, although it can be relevant, for example, to model small superconducting polycrystals. With the help of our analogy, we can derive the field dependence of the Josephson critical current density of three junctions in parallel, a case that is rarely discussed in textbooks.

Before we describe the analogy in detail, we stress that we will introduce two angles. The angle θ is the angle between the pendula, such that a larger θ represents a larger Josephson phase difference between junctions and hence, a stronger applied field.[7] If more than two junctions are involved, it can be shown that the phase difference between two consecutive junctions is identical, and is given by the phase difference between the first and last junctions, $\theta_{n,1}$, divided by the total number of junctions.[3] The angle ϕ is the overall angular rotation of the set of pendula, which we define as the angular deflection relative to the vertical of the first pendulum. Unlike θ, ϕ varies due to the variation in the mass M. This correlation between mass and phase is equivalent to the increase of the superconducting phase in one of the members of a set of superconducting junctions as the applied current is increased.[8]

Let us first consider the case of zero applied magnetic field in the three junction system. In this case, the phase difference between the first and third pendula must be zero (and, of course, also their phase differences relative to the second pendulum). So, we start our system with the three pendula hanging vertically. Then, as mass is added to M, that is, as the current through the junctions is increased, the pendula rotate, that is, the overall phase difference, ϕ, increases and reaches a new equilibrium position. If we keep adding mass,

Table I. Analogies between a Josephson junction and a pendulum (see the text). It can be shown that V is the voltage drop across the junction.

Josephson junction	Pendulum
Phase difference, φ	Deflection, ϕ
Total current, I	Applied torque, Mgr
Josephson current, I_j	$mgL \sin \phi$
$(\hbar/2e)C$	Moment of inertia, Γ
$(\hbar/2e)(1/R)$	Viscous damping, η
Voltage, $V=(\hbar/2e)(d\varphi/dt)$	Angular velocity $\omega=d\phi/dt$

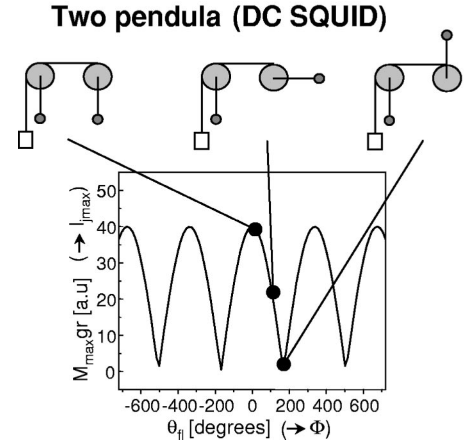

Fig. 2. Calculated maximum mass, $M_{\max}$, vs the phase difference between pendula ($\theta_{n,1}$) corresponding to two Josephson junctions in parallel. $I_{j\max}$ stands for the maximum Josephson current, and Φ for the magnetic flux at the junction (proportional to the applied magnetic field). In the calculations, M is increased until the system starts to rotate by itself: the corresponding mass (analogous to the maximum Josephson current) is defined to be $M_{\max}$. This system reproduces the well-known result for a dc SQUID device.

there will be a maximum value at which any further addition will make the system rotate by itself, that is, $d\phi/dt \neq 0$. This critical mass is able to bring the three pendula to the horizontal position, $\phi_{\max}=90°$. This situation corresponds to a torque of $3mgL$, which represents the maximum Josephson current of the system, $3I_{cj}$.

Let us now consider the behavior for nonzero applied magnetic field for three junctions in parallel. One of the most interesting nonzero field situations for three pendula is represented by the third diagram in Fig. 3, where the first and last pendula are vertically down, while the middle one is vertically up. Observe that the difference between the first and last angle is $\theta_{3,1}=360°$, while the difference between the first and middle pendula is $\theta_{1,2}=360°/2=180°$. For this particular applied magnetic field, the maximum torque is clearly smaller than for zero field, although it corresponds to a local maximum, as we will see.

The idea is that by varying the relative angles between the pendula, we can represent the effect of magnetic fields on arrangements of Josephson junctions. Then, we can calculate or experimentally determine in a simple mechanical fashion the maximum torque of the system for different phase differ-

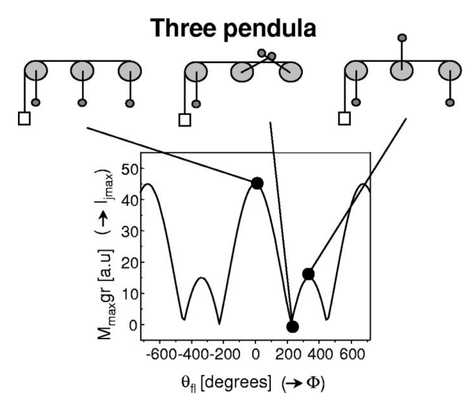

Fig. 3. Same as Fig. 2, but for the case of three Josephson junctions in parallel.

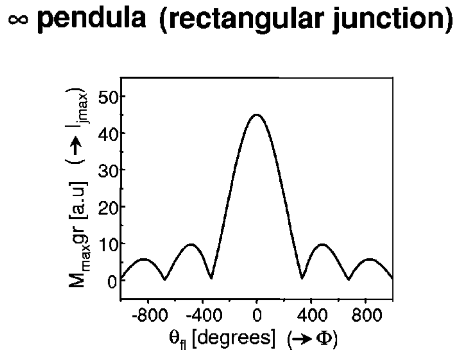

Fig. 4. Same as Fig. 2, but for the case of an infinite set of Josephson junctions in parallel. This system reproduces the Fraunhofer-type-pattern typical of a rectangular Josephson junction.

ences, which is equivalent to the maximum Josephson current versus field characteristics of real Josephson arrangements. As in the zero field case, the maximum torque is simply proportional to the maximum angle, ϕ_{max}, that the system can be deflected without rotating by itself.

The results from theoretical calculations are shown in Figs. 2, 3, and 4 for two, three, and infinite pendula in parallel. Because our goal is to model the magnetic field dependence of the maximum Josephson current, our mechanical problem reduces to finding the maximum mass, M_{max}, that can be added without the system rotating by itself. Following the analogy between the Josephson voltage and the time derivative of ϕ (see Fig. 1), M_{max} is equivalent to the maximum current without voltage drop in the junction. One approach to finding the corresponding equilibrium positions consists of first writing the total potential energy of the system and then minimizing it relative to the deflection of the first pendulum, ϕ_1. (We neglect the friction at the axes of the pulleys.) This approach results in an expression that gives the values of M at the equilibrium positions of the pendula as a function of ϕ_1. We then maximize this function, from which we can obtain M_{max} as a function of the angle between the first and the last pendula, $\theta_{n,1}$. Although this calculation is trivial for two and three pendula, it becomes more complicated for n pendula, particularly for $n \to \infty$.[9] However, our experience is that even the latter case can be understood by undergraduates. The resulting expressions for the phase dependence of the maximum Josephson mass, M_{max}, are given below:

$$M_{max}gr = 2mgL|(\cos(\theta_{2,1}/2))| \quad (n=2), \tag{4}$$

$$M_{max}gr = mgL|(1+2\cos(\theta_{3,1}/2))| \quad (n=3). \tag{5}$$

For an infinite set of parallel junctions, we have

$$M_{max}gr = m_0 gL\left|\frac{\sin(\theta_{n,1}/2)}{\theta_{n,1}/2}\right|. \tag{6}$$

Note that Eqs. (4) and (6) are analogous to the "classical" result for a dc SQUID, $I_{j\,max} \sim |\cos(\pi\Phi/\Phi_0)|$, and to the Fraunhofer-type pattern expected for a rectangular Josephson junction

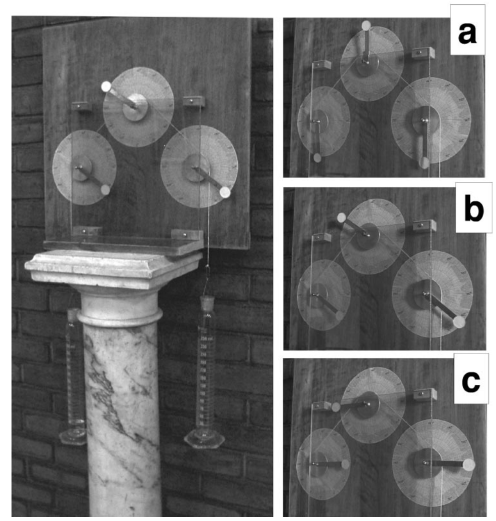

Fig. 5. Apparatus for studying two and three pendula analogs of Josephson junctions in a magnetic field. The three pendula experiment corresponding to the applied field associated with the secondary maximum in the critical current (that is, $\theta_{n,1} = 360°$) is illustrated. The three parts in the right section of the figure illustrate the analogs to (a) $I=0$, (b) $0<I<I_{j\,max}$, and (c) $I \approx I_{j\,max}$.

$$I_{j\,max} \sim \left|\frac{\sin(\pi\Phi/\Phi_0)}{\pi\Phi/\Phi_0}\right|,$$

respectively.[2,3] Φ is the magnetic flux in the junction proportional to the applied magnetic field, and Φ_0 is the flux quantum.[2,3]

Figure 5 shows a set of pendula that lets us study experimentally the case of two and three junctions. (The photo-

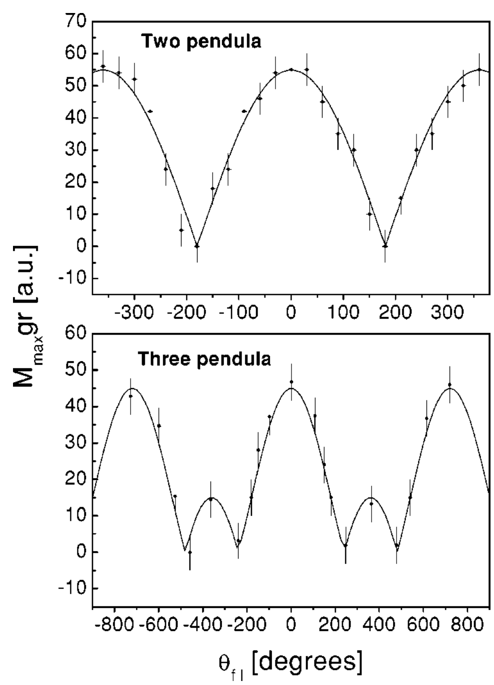

Fig. 6. Comparison between the experimental results obtained with the setup illustrated in Fig. 5 and the theoretical calculations illustrated in Figs. 2 and 3 for two and three pendula. The experimental points are represented by black circles (error bars along x and y are included), and the continuous lines correspond to the theoretical fits of Eqs. (4) and (5).

graph was taken when the system was prepared for measurements of three pendula.) To accommodate the relative phases corresponding to different magnetic fields, we need some tension on the rope before starting to increase M, which we achieve by using a rope with two empty measuring cylinders at each end. Once the desired phases on the pendula are set (making good use of the friction between the pulleys and the rope), we start to add water to the left cylinder. When we reach the maximum amount of water able to maintain a static system, we take note of the corresponding volume in the cylinder, which is proportional to the maximum torque for the corresponding phase difference. Attaining a small friction in the pulley axes was important for reproducing the theoretical expectations. Figure 6 shows the comparison between the experimental results obtained in this way and the theoretical calculations from Eqs. (4) and (5).

III. CONCLUSIONS

We have shown that a set of rigid pendula linked by a common rope reproduces the magnetic field dependence of the Josephson maximum current of two and three Josephson junctions in parallel, and the Fraunhofer-type pattern expected from a rectangular junction. The mechanical analog is easy to setup and work out experimentally, at least in the first two cases, and the theoretical calculations can be performed by elementary methods. The physical insight gained from our mechanical model can be eventually used beyond Josephson phenomena, for example, Young interference and light diffraction by a grating and by a rectangular slit.

Our mechanical model lends itself to problems and projects suitable for student work. For example, calculating $M_{\max}$ vs $\theta_{n,1}$ for two and three pendula, and then comparing the first one with the applied field dependence of $I_{\mathrm{j\,max}}$ for a dc SQUID. An appropriate theoretical project would be to extend these calculations to an infinite set of pendula, and compare it with a rectangular Josephson junction. A feasible experimental project would be to construct a mechanical analog to the one depicted in Fig. 5 to experimentally reproduce the $M_{\max}$ vs $\theta_{n,1}$ characteristics for one, two, and three pendula. It is important to stress that the friction at the pulleys axes must be small.

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[7]See, for example, Eq. (8.68) for two junctions in Ref. 3.

[8]See, for example, Eq. (8.69) for two junctions in Ref. 3.

[9]A necessary physical assumption in this case is that the sum of the masses has some finite value, m_0. It is equivalent to assuming that we can split a rectangular Josephson junction into an infinite set of small junctions in parallel, taking into account that the Josephson critical current of the whole system remains finite.